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BÄCKLUND TRANSFORMATIONS FOR FOURTH PAINLEVÉ HIERARCHIES

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Abstract. Bäcklund transformations (BTs) for ordinary differential equations (ODEs), and in particular for hierarchies of ODEs, are a topic of great current interest. Here we give an improved method of constructing BTs for hierarchies of ODEs. This approach is then applied to fourth Painlevé (P_{IV}) hierarchies recently found by the same authors [*Publ. Res. Inst. Math. Sci. (Kyoto)* **37** 327–347 (2001)]. We show how the known pattern of BTs for P_{IV} can be extended to our P_{IV} hierarchies. Remarkably, the BTs required to do this are precisely the Miura maps of the dispersive water wave hierarchy. We also obtain the important result that the fourth Painlevé equation has only one nontrivial fundamental BT, and not two such as is frequently stated.

1. Introduction

A classical problem, dating from the end of the nineteenth century, is that of seeking new transcendental functions defined by ordinary differential equations (ODEs). This motivated the classification of ODEs having what is today referred to as the Painlevé property, i.e. having their general solution free of movable branched singularities. In particular, it led to the discovery of the six Painlevé equations [1, 2, 3, 4], which did indeed define new transcendental functions.

The six Painlevé equations are of course second order ODEs. However the classification program embarked upon by Painlevé and co-workers foresaw, once second order ODEs had been dealt with, a classification of third order ODEs, then of fourth order ODEs, and so on. Thus Chazy [5] and Garnier [6] studied certain classes of third order ODEs, although no new transcendent was discovered at third order. Restricted classes of third order ODEs were also later considered by Exton [7] and Martynov [8, 9], unfortunately with the same result. It should be remarked that the difficulties of classification increase with the order of the equations studied; for example, at second order movable essential singularities may arise [2], whereas at third order movable natural boundaries may occur [5]. At fourth order, even the classification of dominant terms for the polynomial case was left incomplete [10].

Thus, some 20-25 years ago, the search for higher order ODEs defining new transcendental functions was in need of a new insight in order to catalyse research in

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this area. This impetus came in the form of the discovery by Ablowitz and Segur [11] of a connection between completely integrable partial differential equations (PDEs) and ODEs having the Painlevé property. This discovery not only made a remarkable connection between modern research and mathematics at the turn of the last century, but in establishing a link between integrability and the analytical properties of solutions, mirrored the prize-winning work of Kowalevski on the motion of a rigid body about a fixed point [12, 13]. It was Airault who, exploiting the fact that, for example, sitting above the Korteweg-de Vries (KdV) equation and the modified KdV (mKdV) equation are their respective hierarchies, first realised the next step of using higher order integrable PDEs to derive higher order ODEs with the Painlevé property. In fact she derived a whole hierarchy of ODEs, the second Painlevé (P_{II}) hierarchy, by similarity reduction of the KdV/mKdV hierarchies [14].

However Airault also made another important step: she obtained Bäcklund transformations (BTs) for every member of the P_{II} hierarchy. A BT is a mapping between solutions of ODEs, involving naturally some identification between the parameters appearing as coefficients in the ODEs; in the case of BTs between solutions of the same ODE, this identification between parameters translates as changes in paramater values. BTs for the Painlevé equations had previously been studied in the Soviet literature; a comprehensive list of references can be found in [15], and a recent review in [16]. Today BTs are universally recognised as an important property of integrable nonlinear ODEs, and there is much interest in their derivation, especially within the context of hierarchies of ODEs. The aim of the present paper is to explore BTs for fourth Painlevé (P_{IV}) hierarchies.

Due therefore to the work of Ablowitz and Segur, over the last quarter century, the study of higher order analogues of the Painlevé equations, and of their properties, has been informed by knowledge of the connection with completely integrable PDEs; here we refer for example to the work of Muğan and Jrad [17, 18, 19], and Cosgrove [20]. The present authors have also exploited this connection [21, 22, 23, 24] in the development of their own method [22] of deriving (amongst other things) hierarchies of higher order Painlevé equations together with associated underlying linear problems. Here we extend this connection still further: we find that certain features of such ODEs are directly related to the underlying structures of associated completely integrable PDEs. That is, when seeking to extend to a fourth Painlevé hierarchy [24] the pattern of BTs already known for the first member (P_{IV}), we find that the answer lies in the Miura transformations for the associated PDE hierarchy.

The layout of the paper is as follows. We introduce our P_{IV} hierarchies in Section Two. In Section Three we give an improved method, based on the Painlevé truncation process for PDEs, of deriving auto-BTs and special integrals for hierarchies of ODEs, and as an example we apply this approach to the P_{II} hierarchy of Airault. In Section Four we use this method to derive auto-BTs and special integrals for two of the P_{IV} hierarchies derived in Section Two. In Section Five we identify to which BTs of P_{IV} these BTs correspond. In Section Six we seek further BTs in order to extend the known pattern of BTs for P_{IV} to corresponding hierarchies. Remarkably, it turns out that the BTs required to do this are precisely the known Miura maps for the associated PDE (dispersive water wave) hierarchy. In Section Seven we consider a mapping between our hierarchies which allows us to further relate the BTs derived: an important consequence of this is the result that P_{IV} has only one nontrivial fundamental BT. Section Eight is devoted to conclusions.

2. Sequences of fourth Painlevé hierarchies

In our recent paper [24] we derived, along with associated linear problems, the sequence of coupled ODEs in $\mathbf{u} = (u, v)^T$,

$$\mathcal{R}^n \mathbf{u}_x + \sum_{i=0}^{n-1} c_i \mathcal{R}^i \mathbf{u}_x + g_{n-1}\mathcal{R}^2 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + g_n \mathcal{R} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + g_{n+1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{1}$$

where $c_0, \ldots, c_{n-1}$, g_{n-1}, g_n and g_{n+1}, are arbitrary constants, and $\mathcal{R}$ is the recursion operator of the dispersive water wave (DWW) hierarchy [25]—[32] ($\partial_x = \partial/\partial x = d/dx$ in our ODE case (1)),

$$\mathcal{R} = \frac{1}{2} \begin{pmatrix} \partial_x u \partial_x^{-1} - \partial_x & 2 \\ 2v + v_x \partial_x^{-1} & u + \partial_x \end{pmatrix}. \tag{2}$$

In what follows we will consider the case which corresponds to a generalized P_{IV} hierarchy, i.e. $g_{n-1} = 0$ and $g_n \neq 0$ [24]. We can then assume, using a shift on u, that $g_{n+1} = 0$ (note that previously we have used such a shift to set $c_{n-1} = 0$, but here we prefer to remove g_{n+1}). Further, using a shift on x, we can set $c_0 = 0$. Thus, without any loss of generality, we can assume that our generalized P_{IV} hierarchy is of the form

$$\mathcal{R}^n \mathbf{u}_x + \sum_{i=1}^{n-1} h_i \mathcal{R}^i \mathbf{u}_x + g_n \mathcal{R} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{3}$$

for some constants $h_1, \ldots, h_{n-1}$ and $g_n (\neq 0)$. We note in passing that the second nontrivial member of our hierarchy ($n = 2$) is of interest for the problems that its singularity analysis presents; this was the subject of our paper [33].

The hierarchy (3) can also be written in the alternative form

$$B_2 \mathbf{K}_n[\mathbf{u}] = 0, \tag{4}$$

where

$$\mathbf{K}_n[\mathbf{u}] = \mathbf{L}_n[\mathbf{u}] + \sum_{i=1}^{n-1} h_i \mathbf{L}_i[\mathbf{u}] + g_n \begin{pmatrix} 0 \\ x \end{pmatrix}, \tag{5}$$

B_2 is one of the three Hamiltonian operators of the DWW hierarchy,

$$B_2 = \frac{1}{2} \begin{pmatrix} 2\partial_x & \partial_x u - \partial_x^2 \\ u\partial_x + \partial_x^2 & v\partial_x + \partial_x v \end{pmatrix}, \tag{6}$$

and each $\mathbf{L}_i[\mathbf{u}]$ is the variational derivative of the Hamiltonian density corresponding to the operator B_2 for the t_i-flow of the DWW hierarchy, $\mathbf{u}_{t_i} = \mathcal{R}^i \mathbf{u}_x = B_2 \mathbf{L}_i[\mathbf{u}]$.

Here we have used the fact that $\mathcal{R} = B_2 B_1^{-1}$, where

$$B_1 = \begin{pmatrix} 0 & \partial_x \\ \partial_x & 0 \end{pmatrix} \tag{7}$$

is another of the Hamiltonian operators of the DWW hierarchy. We note also the recursion relation $B_1 \mathbf{L}_{i+1}[\mathbf{u}] = B_2 \mathbf{L}_i[\mathbf{u}]$, and that $\mathbf{L}_0[\mathbf{u}] = (0, 2)^T$, $\mathbf{L}_1[\mathbf{u}] = (v, u)^T$.

We now consider the construction of hierarchies equivalent to (4). We will also see how a reduction of order of our system (4) can be effected using the Hamiltonian structures of the DWW hierarchy. We begin by recalling the Miura maps of the

DWW hierarchy, as given by Kupershmidt [31]. The first Miura map is given by $\mathbf{u} = \mathbf{F}[\mathbf{U}]$, where $\mathbf{U} = (U, V)^T$ and

$$\mathbf{F}[\mathbf{U}] = \begin{pmatrix} U \\ UV - V^2 + V_x \end{pmatrix}. \tag{8}$$

The two second Miura maps are given by $\mathbf{U} = \mathbf{\Phi}[\boldsymbol{\phi}]$, where $\boldsymbol{\phi} = (\phi, p)^T$ and

$$\mathbf{\Phi}[\boldsymbol{\phi}] = \begin{pmatrix} \phi + 2p \\ p \end{pmatrix}, \tag{9}$$

and $\mathbf{U} = \mathbf{\Psi}[\boldsymbol{\psi}]$, where $\boldsymbol{\psi} = (\psi, s)^T$ and

$$\mathbf{\Psi}[\boldsymbol{\psi}] = \begin{pmatrix} \psi - 2s \\ -s \end{pmatrix}. \tag{10}$$

That is, we have the following two sequences of Miura transformations:

$$\begin{tikzcd} \begin{pmatrix} u \\ v \end{pmatrix} \arrow[r, "\mathbf{F}"'] & \begin{pmatrix} U \\ V \end{pmatrix} \arrow[r, "\mathbf{\Phi}"'] \arrow[dr, "\mathbf{\Psi}"'] & \begin{pmatrix} \phi \\ p \end{pmatrix} \\ & & \begin{pmatrix} \psi \\ s \end{pmatrix} \end{tikzcd} \tag{11}$$

We now consider the first of these sequences. Using the fact that for the Miura map $\mathbf{F}$ we have

$$B_2 \Big|_{\mathbf{u}=\mathbf{F}[\mathbf{U}]} = \mathbf{F}'[\mathbf{U}] B \left(\mathbf{F}'[\mathbf{U}]\right)^\dagger, \tag{12}$$

where B is the Hamiltonian operator of the modified DWW system,

$$B = \frac{1}{2} \begin{pmatrix} 2\partial_x & \partial_x \\ \partial_x & 0 \end{pmatrix}, \tag{13}$$

$\mathbf{F}'[\mathbf{U}]$ is the Fréchet derivative of the Miura map and $(\mathbf{F}'[\mathbf{U}])^\dagger$ is its adjoint, we obtain, in the same way as in the PDE case, the modified version of (4),

$$B \left(\mathbf{F}'[\mathbf{U}]\right)^\dagger \mathbf{K}_n[\mathbf{F}[\mathbf{U}]] = 0. \tag{14}$$

This we can then integrate to obtain

$$\left(\mathbf{F}'[\mathbf{U}]\right)^\dagger \mathbf{K}_n[\mathbf{F}[\mathbf{U}]] + (c_n, d_n)^T = 0, \tag{15}$$

for two arbitrary constants c_n and d_n. In fact this last is equivalent to an integrated version of (4) under the BT

$$\begin{aligned} \mathbf{u} - \mathbf{F}[\mathbf{U}] &= 0, & (16)\\ \left(\mathbf{F}'[\mathbf{U}]\right)^\dagger \mathbf{K}_n[\mathbf{u}] + (c_n, d_n)^T &= 0. & (17) \end{aligned}$$

It is this construction of a BT between an integrated modified hierarchy and an integrated version of our original hierarchy that lies behind the first integrals of our P_{IV} hierarchy given in [24]; this approach is described in more detail in [34]. The integrated form of (4) obtained from (16), (17) is

$$\begin{aligned} L_{n,x} &= 2K_n + uL_n + (g_n - 2\alpha_n), & (18)\\ K_{n,x} &= \frac{(K_n + \frac{1}{2}g_n - \alpha_n)^2 - \frac{1}{4}\beta_n^2}{L_n} - vL_n, & (19) \end{aligned}$$

where $\mathbf{K}_n = (K_n, L_n)^T$, and where we have set $2c_n + d_n = g_n - 2\alpha_n$ and $d_n^2 = \beta_n^2$.

In the same way, since under the composition $\mathbf{H} = \mathbf{F} \circ \mathbf{\Phi}$ we have analogously to (12)

$$B_2 \Big|_{\mathbf{u}=\mathbf{H}[\boldsymbol{\phi}]} = \mathbf{H}'[\boldsymbol{\phi}] C \left(\mathbf{H}'[\boldsymbol{\phi}]\right)^{\dagger} \tag{20}$$

with

$$C = \frac{1}{2}\begin{pmatrix} -2\partial_x & \partial_x \\ \partial_x & 0 \end{pmatrix}, \tag{21}$$

we obtain the integrated second modified hierarchy,

$$\left(\mathbf{H}'[\boldsymbol{\phi}]\right)^{\dagger} \mathbf{K}_n[\mathbf{H}[\boldsymbol{\phi}]] + (e_n, f_n)^T = 0. \tag{22}$$

It is easy to see that the constants of integration in (15) and (22) are related by $c_n = e_n$ and $d_n = f_n - 2e_n$.

For our second sequence of Miura transformations we have the composition $\mathbf{I} = \mathbf{F} \circ \mathbf{\Psi}$ and, corresponding to (20),

$$B_2 \Big|_{\mathbf{u}=\mathbf{I}[\boldsymbol{\psi}]} = \mathbf{I}'[\boldsymbol{\psi}] D \left(\mathbf{I}'[\boldsymbol{\psi}]\right)^{\dagger} \tag{23}$$

with

$$D = \frac{1}{2}\begin{pmatrix} -2\partial_x & -\partial_x \\ -\partial_x & 0 \end{pmatrix}. \tag{24}$$

Thus we obtain the alternative integrated second modified hierarchy,

$$\left(\mathbf{I}'[\boldsymbol{\psi}]\right)^{\dagger} \mathbf{K}_n[\mathbf{I}[\boldsymbol{\psi}]] + (l_n, m_n)^T = 0, \tag{25}$$

with constants of integration related to those of (15) by $c_n = l_n$ and $d_n = -m_n - 2l_n$.

The hierarchies (18)—(19), (15), (22) and (25) are all P_{IV} hierarchies. In order to show this, let us consider the case $n = 1$ of these hierarchies. We have $K_1 = v$ and $L_1 = u + g_1 x$, and so our system (18)—(19) reads

$$u_x = 2v + u(u + g_1 x) - 2\alpha_1, \tag{26}$$

$$v_x = \frac{(v + \frac{1}{2}g_1 - \alpha_1)^2 - \frac{1}{4}\beta_1^2}{u + g_1 x} - v(u + g_1 x); \tag{27}$$

eliminating v and setting $u = \pm y - g_1 x$ yields the fourth Painlevé equation,

$$y_{xx} = \frac{1}{2}\frac{y_x^2}{y} + \frac{3}{2}y^3 \mp 2g_1 x y^2 + 2\left(\frac{1}{4}g_1^2 x^2 - \alpha_1\right) y - \frac{1}{2}\frac{\beta_1^2}{y}. \tag{28}$$

The system (15) reads

$$V_x + 2UV - V^2 + g_1 x V + c_1 = 0, \tag{29}$$

$$U_x + 2UV - U^2 - g_1(U - 2V)x + g_1 - d_1 = 0. \tag{30}$$

Elimination of V and setting $U = \pm y - g_1 x$, $2c_1 + d_1 = g_1 - 2\alpha_1$ and $d_1^2 = \beta_1^2$ yields (28). This follows immediately from the fact that in the Miura map $U = u$. However, eliminating U and setting $V = \pm w$ also yields the fourth Painlevé equation,

$$w_{xx} = \frac{1}{2}\frac{w_x^2}{w} + \frac{3}{2}w^3 \pm 2g_1 x w^2 + 2\left[\frac{1}{4}g_1^2 x^2 - \frac{1}{2}(c_1 + 2d_1 - g_1)\right] w - \frac{1}{2}\frac{c_1^2}{w}. \tag{31}$$

Thus for $n = 1$, both independent variables of the first modification define versions of P_{IV}. We will return to the relationship between these two copies of P_{IV} later.

Our first second modification (22), for $n = 1$, reads

$$p_x + 2\phi p + 3p^2 + g_1 xp + e_1 = 0, \tag{32}$$

$$\phi_x - 6\phi p - 6p^2 - \phi^2 - g_1 x(\phi + 2p) + g_1 - f_1 = 0; \tag{33}$$

eliminating ϕ yields

$$p_{xx} = \frac{1}{2}\frac{p_x^2}{p} + \frac{3}{2}p^3 + 2g_1 xp^2 + 2\left[\frac{1}{4}g_1^2x^2 - \frac{1}{2}(2f_1 - 3e_1 - g_1)\right]p - \frac{1}{2}\frac{e_1^2}{p}, \tag{34}$$

i.e. the fourth Painlevé equation. Noting that in the Miura map $V = p$, we see that this last is equivalent to (31), for the upper choice of sign, with the identification $w = p$, $c_1 = e_1$ and $d_1 = f_1 - 2e_1$.

Our second second modification (25), for $n = 1$, reads

$$s_x + 2\psi s - 3s^2 + g_1 xs - l_1 = 0, \tag{35}$$

$$\psi_x + 6\psi s - 6s^2 - \psi^2 - g_1 x(\psi - 2s) + g_1 + m_1 = 0, \tag{36}$$

equivalent to (32), (33) under $(\phi, p, e_1, f_1) \to (\psi, -s, l_1, -m_1)$. Elimination of ψ gives

$$s_{xx} = \frac{1}{2}\frac{s_x^2}{s} + \frac{3}{2}s^3 - 2g_1 xs^2 + 2\left[\frac{1}{4}g_1^2x^2 + \frac{1}{2}(2m_1 + 3l_1 + g_1)\right]s - \frac{1}{2}\frac{l_1^2}{s}, \tag{37}$$

another version of P_{IV} equivalent to (31) for the lower choice of sign, with $w = s$, $c_1 = l_1$ and $d_1 = -m_1 - 2l_1$.

Thus we see that the hierarchies (18)—(19), (15), (22) and (25) define sequences of P_{IV} hierarchies, as in (11). In Section Four we will derive BTs for the hierarchies (22) and (25), and show in Section Six how the known structure of BTs for P_{IV} can be replicated for P_{IV} hierarchies, using the Miura transformations given above.

Before turning to the derivation of BTs and special integrals for P_{IV} hierarchies, however, we present first of all an improved method of deriving BTs for hierarchies of ODEs. As a simple but illuminating example, we apply this to the P_{II} hierarchy.

3. Bäcklund transformations for the second Painlevé hierarchy

We take the P_{II} hierarchy in the form

$$(\partial_x + 2Y)\left(M_n[Y_x - Y^2] - \frac{1}{2}x\right) + \frac{1}{2} - \lambda_n = 0, \tag{38}$$

where λ_n are arbitrary parameters, and the sequence M_n satisfies the Lenard recursion relation [35] $\partial_x M_{n+1}[W] = (\partial_x^3 + 4W\partial_x + 2W_x)M_n[W]$, with $M_0 = 1/2$, $M_1[W] = W$. In order to construct a BT for this hierarchy, we consider adapting the approach developed by Weiss for PDEs [36], and seek a "truncated Painlevé expansion"

$$Y = -\frac{\varphi_x}{\varphi} + \tilde{Y}, \tag{39}$$

where

$$\tilde{Y} = \frac{1}{2}\frac{\varphi_{xx}}{\varphi_x}. \tag{40}$$

For Y defined by (39), we find that

$$Y_x - Y^2 = \tilde{Y}_x - \tilde{Y}^2 - \frac{\varphi_{xx}}{\varphi} + 2\frac{\varphi_x}{\varphi}\tilde{Y} = \tilde{Y}_x - \tilde{Y}^2, \tag{41}$$

where in order to obtain the last equality we have used (40). That is, the quantity $Y_x - Y^2$ is invariant under the mapping (39), (40). Thus substituting (39) into (38) yields

$$\left(\partial_x + 2\tilde{Y} - 2\frac{\varphi_x}{\varphi}\right)\left(M_n[\tilde{Y}_x - \tilde{Y}^2] - \frac{1}{2}x\right) + \frac{1}{2} - \lambda_n = 0, \tag{42}$$

Assuming now that $\tilde{Y}$ also satisfies the corresponding member of the P_{II} hierarchy, but now for parameter value $\tilde{\lambda}_n$, i.e.

$$(\partial_x + 2\tilde{Y})\left(M_n[\tilde{Y}_x - \tilde{Y}^2] - \frac{1}{2}x\right) + \frac{1}{2} - \tilde{\lambda}_n = 0, \tag{43}$$

we obtain using (42) and this last,

$$\frac{\varphi_x}{\varphi} = \frac{\tilde{\lambda}_n - \lambda_n}{2M_n[\tilde{Y}_x - \tilde{Y}^2] - x}. \tag{44}$$

But (44) must be compatible with (40), or equivalently with the Riccati equation

$$\left(\frac{\varphi_x}{\varphi}\right)_x + \left(\frac{\varphi_x}{\varphi}\right)^2 - 2\tilde{Y}\left(\frac{\varphi_x}{\varphi}\right) = 0; \tag{45}$$

substituting (44) in (45) gives

$$(\partial_x + 2\tilde{Y})\left(M_n[\tilde{Y}_x - \tilde{Y}^2] - \frac{1}{2}x\right) + \frac{1}{2}(\lambda_n - \tilde{\lambda}_n) = 0, \tag{46}$$

and so comparing with (43) we see that this compatibility requires

$$\lambda_n + \tilde{\lambda}_n = 1. \tag{47}$$

Thus we obtain Airault's BT [14]

$$Y = \tilde{Y} + \frac{\tilde{\lambda}_n - \lambda_n}{x - 2M_n[\tilde{Y}_x - \tilde{Y}^2]} \tag{48}$$

for the P_{II} hierarchy, along with the shift in paramaters (47). We note that this derivation, which does not make use of the Schwarzian derivative but instead relies on the invariance of the quantity $Y_x - Y^2$ under the mapping (39), (40), is much simpler, and is much more widely applicable, than that presented in [37].

Special integrals of the P_{II} hierarchy are obtained by setting coefficients of different powers of φ in (42) to zero independently; since $\tilde{Y}_x - \tilde{Y}^2 = Y_x - Y^2$ we see that this gives

$$M_n[Y_x - Y^2] - \frac{1}{2}x = 0, \tag{49}$$

which defines solutions of (38) for $\lambda_n = 1/2$. We refer to [37] for further information on special integrals of the P_{II} hierarchy, and the iteration of P_{II} hierarchy BTs.

4. Bäcklund transformations for fourth Painlevé hierarchies

We now apply the above approach to our P_{IV} hierarchy (22); since

$$\mathbf{H}[\boldsymbol{\phi}] = \begin{pmatrix} \phi + 2p \\ \phi p + p^2 + p_x \end{pmatrix}, \tag{50}$$

this reads

$$\left(\begin{array}{cc} 1 & p \\ 2 & \phi+2p-\partial_x \end{array}\right)\mathbf{K}_n\left[\left(\begin{array}{c} \phi+2p \\ \phi p+p^2+p_x \end{array}\right)\right]+\left(\begin{array}{c} e_n \\ f_n \end{array}\right)=\left(\begin{array}{c} 0 \\ 0 \end{array}\right). \tag{51}$$

We now seek, analogously to the case of the P_{II} hierarchy above, a mapping (BT) between two solutions ϕ, p and $\tilde{\phi}$, $\tilde{p}$ of our P_{IV} hierarchy, of the form

$$\phi = 2\frac{\varphi_x}{\varphi}+\tilde{\phi}, \tag{52}$$

$$p = -\frac{\varphi_x}{\varphi}+\tilde{p}, \tag{53}$$

where

$$\tilde{\phi} = -\frac{\varphi_{xx}}{\varphi_x}. \tag{54}$$

It then follows that

$$\phi+2p=\tilde{\phi}+2\tilde{p} \tag{55}$$

and

$$\phi p+p^2+p_x=\tilde{\phi}\tilde{p}+\tilde{p}^2+\tilde{p}_x-\frac{\varphi_{xx}}{\varphi}-\frac{\varphi_x}{\varphi}\tilde{\phi}=\tilde{\phi}\tilde{p}+\tilde{p}^2+\tilde{p}_x, \tag{56}$$

where the last equality follows from (54). Thus we see that the quantities $\phi+2p$ and $\phi p+p^2+p_x$ are invariant under the mapping (52), (53), (54). Substitution of (52), (53) into (51) therefore gives

$$\left(\begin{array}{cc} 1 & \tilde{p}-\frac{\varphi_x}{\varphi} \\ 2 & \tilde{\phi}+2\tilde{p}-\partial_x \end{array}\right)\mathbf{K}_n\left[\left(\begin{array}{c} \tilde{\phi}+2\tilde{p} \\ \tilde{\phi}\tilde{p}+\tilde{p}^2+\tilde{p}_x \end{array}\right)\right]+\left(\begin{array}{c} e_n \\ f_n \end{array}\right)=\left(\begin{array}{c} 0 \\ 0 \end{array}\right). \tag{57}$$

Since we assume that $\tilde{\phi}$, $\tilde{p}$ are solutions of a second copy of our P_{IV} hierarchy, but with parameters $\tilde{e}_n$, $\tilde{f}_n$, i.e.

$$\left(\begin{array}{cc} 1 & \tilde{p} \\ 2 & \tilde{\phi}+2\tilde{p}-\partial_x \end{array}\right)\mathbf{K}_n\left[\left(\begin{array}{c} \tilde{\phi}+2\tilde{p} \\ \tilde{\phi}\tilde{p}+\tilde{p}^2+\tilde{p}_x \end{array}\right)\right]+\left(\begin{array}{c} \tilde{e}_n \\ \tilde{f}_n \end{array}\right)=\left(\begin{array}{c} 0 \\ 0 \end{array}\right), \tag{58}$$

we obtain, by elimination between (57) and this last,

$$\frac{\varphi_x}{\varphi} = \frac{e_n-\tilde{e}_n}{L_n\left[\left(\begin{array}{c} \tilde{\phi}+2\tilde{p} \\ \tilde{\phi}\tilde{p}+\tilde{p}^2+\tilde{p}_x \end{array}\right)\right]}, \tag{59}$$

$$\tilde{f}_n = f_n. \tag{60}$$

Equation (59) must be compatible with (54), or equivalently with the Riccati equation

$$\left(\frac{\varphi_x}{\varphi}\right)_x+\left(\frac{\varphi_x}{\varphi}\right)^2+\tilde{\phi}\left(\frac{\varphi_x}{\varphi}\right)=0; \tag{61}$$

substituting (59) into (61) gives

$$(\tilde{\phi}-\partial_x)L_n\left[\left(\begin{array}{c} \tilde{\phi}+2\tilde{p} \\ \tilde{\phi}\tilde{p}+\tilde{p}^2+\tilde{p}_x \end{array}\right)\right]+(e_n-\tilde{e}_n)=0, \tag{62}$$

and comparing this last with (58) we see that we must have $\tilde{e}_n=\tilde{f}_n-e_n$ and so

$$\tilde{e}_n=f_n-e_n. \tag{63}$$

Thus we have for our P_{IV} hierarchy (51) the BT

$$\phi = \tilde{\phi} + 2\frac{e_n - \tilde{e}_n}{L_n\left[\begin{pmatrix} \tilde{\phi} + 2\tilde{p} \\ \tilde{\phi}\tilde{p} + \tilde{p}^2 + \tilde{p}_x \end{pmatrix}\right]}, \tag{64}$$

$$p = \tilde{p} - \frac{e_n - \tilde{e}_n}{L_n\left[\begin{pmatrix} \tilde{\phi} + 2\tilde{p} \\ \tilde{\phi}\tilde{p} + \tilde{p}^2 + \tilde{p}_x \end{pmatrix}\right]}, \tag{65}$$

along with the shifts in parameters given by (60) and (63).

We now consider deriving BTs for the P_{IV} hierarchy (25); since

$$\mathbf{I}[\boldsymbol{\psi}] = \begin{pmatrix} \psi - 2s \\ -\psi s + s^2 - s_x \end{pmatrix}, \tag{66}$$

this reads

$$\begin{pmatrix} 1 & -s \\ -2 & -\psi + 2s + \partial_x \end{pmatrix} \mathbf{K}_n\left[\begin{pmatrix} \psi - 2s \\ -\psi s + s^2 - s_x \end{pmatrix}\right] + \begin{pmatrix} l_n \\ m_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{67}$$

Seeking a BT in the form

$$\psi = 2\frac{\varphi_x}{\varphi} + \hat{\psi}, \tag{68}$$

$$s = \frac{\varphi_x}{\varphi} + \hat{s}, \tag{69}$$

where

$$\hat{\psi} = -\frac{\varphi_{xx}}{\varphi_x}, \tag{70}$$

and where $\hat{\psi}$, $\hat{s}$ are solutions of our P_{IV} hierarchy for parameter values $\hat{l}_n$, $\hat{m}_n$,

$$\begin{pmatrix} 1 & -\hat{s} \\ -2 & -\hat{\psi} + 2\hat{s} + \partial_x \end{pmatrix} \mathbf{K}_n\left[\begin{pmatrix} \hat{\psi} - 2\hat{s} \\ -\hat{\psi}\hat{s} + \hat{s}^2 - \hat{s}_x \end{pmatrix}\right] + \begin{pmatrix} \hat{l}_n \\ \hat{m}_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{71}$$

then yields

$$\psi = \hat{\psi} + 2\frac{l_n - \hat{l}_n}{L_n\left[\begin{pmatrix} \hat{\psi} - 2\hat{s} \\ -\hat{\psi}\hat{s} + \hat{s}^2 - \hat{s}_x \end{pmatrix}\right]}, \tag{72}$$

$$s = \hat{s} + \frac{l_n - \hat{l}_n}{L_n\left[\begin{pmatrix} \hat{\psi} - 2\hat{s} \\ -\hat{\psi}\hat{s} + \hat{s}^2 - \hat{s}_x \end{pmatrix}\right]}, \tag{73}$$

for the shift in parameter values

$$\hat{m}_n = m_n, \tag{74}$$

$$\hat{l}_n = -m_n - l_n. \tag{75}$$

We note that the BT (72)—(75) follows immediately from (64), (65), (60), (63) under $(\phi, p, e_n, f_n) \to (\psi, -s, l_n, -m_n)$, which maps the P_{IV} hierarchy (51) into the P_{IV} hierarchy (67). However, this mapping does not leave the P_{IV} equation in standard form, since (34) is mapped to (37), and these two equations we identify with P_{IV} by setting $g_1 = 2$ and $g_1 = -2$ respectively. We return to this point later.

We now briefly consider special integrals. We see that setting coefficients of different powers of φ in (57) to zero independently gives, using the fact that $\tilde{\phi}+2\tilde{p} = \phi + 2p$ and $\tilde{\phi}\tilde{p} + \tilde{p}^2 + \tilde{p}_x = \phi p + p^2 + p_x$,

$$L_n \left[\left(\begin{array}{c} \phi + 2p \\ \phi p + p^2 + p_x \end{array} \right) \right] = 0, \tag{76}$$

which then defines solutions of (51) provided that

$$K_n \left[\left(\begin{array}{c} \phi + 2p \\ \phi p + p^2 + p_x \end{array} \right) \right] + e_n = 0 \tag{77}$$

and

$$f_n = 2e_n. \tag{78}$$

In the same way, at the same point in the derivation of the BT (72)—(75), setting coefficients of different powers of φ to zero independently, and using the fact that $\hat{\psi} - 2\hat{s} = \psi - 2s$ and $-\hat{\psi}\hat{s} + \hat{s}^2 - \hat{s}_x = -\psi s + s^2 - s_x$, gives

$$L_n \left[\left(\begin{array}{c} \psi - 2s \\ -\psi s + s^2 - s_x \end{array} \right) \right] = 0, \tag{79}$$

which then defines solutions of (67) provided that

$$K_n \left[\left(\begin{array}{c} \psi - 2s \\ -\psi s + s^2 - s_x \end{array} \right) \right] + l_n = 0 \tag{80}$$

and

$$m_n = -2l_n. \tag{81}$$

Again, just as for our BTs, we have the mapping $(\phi, p, e_n, f_n) \to (\psi, -s, l_n, -m_n)$ between these special integrals. See however the discussion in the next Section.

5. Identification of Bäcklund transformations

We now turn to the identification of the BTs obtained in the previous section. We will give to BTs for our P_{IV} hierarchies the same names as are given to the case $n = 1$, i.e. to the P_{IV} equation itself. First of all we fix the identification of parameters in our hierachies with the parameters α and β in P_{IV} when written as

$$Q_{xx} = \frac{1}{2}\frac{Q_x^2}{Q} + \frac{3}{2}Q^3 + 4xQ^2 + 2\left(x^2 - \alpha\right) Q - \frac{1}{2}\frac{\beta^2}{Q}. \tag{82}$$

We take this last as the standard form of P_{IV} in order to simplify the writing of parameter shifts for its BTs. We note that, since $\beta \to -\beta$ is a discrete symmetry of (82), in BTs of P_{IV} we can always replace parameters corresponding to β by $\pm\beta$.

We begin with the hierarchy (22), or (51),

$$\left(\mathbf{H}'[\boldsymbol{\phi}]\right)^{\dagger} \mathbf{K}_n[\mathbf{H}[\boldsymbol{\phi}]] + (e_n, f_n)^T = 0. \tag{83}$$

In the case $n = 1$ this gives the system (32), (33) and, after eliminating ϕ, equation (34). In order to identify this last with equation (82) we now set, for the entire hierarchy (83),

$$g_n = 2, \tag{84}$$

$$e_n = B_n, \tag{85}$$

$$f_n = \frac{1}{2}(2A_n + 3B_n + 2), \tag{86}$$

and similarly for the parameters $\tilde{e}_n$ and $\tilde{f}_n$ in the hierarchy (58).

We then have for the hierarchy (83) the BT (64), (65), with the corresponding shift on parameters (60), (63), i.e.

$$\tilde{A}_n = -\frac{1}{4}(2A_n - 3B_n + 6), \tag{87}$$

$$\tilde{B}_n = \frac{1}{2}(2A_n + B_n + 2). \tag{88}$$

In the case $n = 1$ this BT reads

$$\phi = \tilde{\phi} + \frac{B_1 - 2A_1 - 2}{\tilde{\phi} + 2\tilde{p} + 2x}, \tag{89}$$

$$p = \tilde{p} - \frac{1}{2}\frac{B_1 - 2A_1 - 2}{\tilde{\phi} + 2\tilde{p} + 2x}. \tag{90}$$

Eliminating $\tilde{\phi}$ we obtain a BT for P_{IV} (34) itself,

$$p = \tilde{p} + \frac{(B_1 - 2A_1 - 2)\tilde{p}}{\tilde{p}_x - \tilde{p}^2 - 2x\tilde{p} + B_1/2 + A_1 + 1}. \tag{91}$$

This BT for P_{IV}, along with the parameter shift (87), (88) (for $n = 1$), is often referred to as the "double dagger" ($t^{\ddagger}$) BT (see [38, 39]). For this reason we refer to the BT (64), (65), together with the parameter shifts (87), (88), as the $t^{\ddagger}$ BT for the P_{IV} hierarchy (83).

We now turn to the hierarchy (25), or (67),

$$\left(\mathbf{I}'[\boldsymbol{\psi}]\right)^{\dagger}\mathbf{K}_n[\mathbf{I}[\boldsymbol{\psi}]] + (l_n, m_n)^T = 0, \tag{92}$$

In the case $n = 1$ this gives the system (35), (36) and, after eliminating ψ, equation (37). In order to identify equation (37) with (82) we now set, for the entire hierarchy (92),

$$g_n = -2, \tag{93}$$

$$l_n = b_n, \tag{94}$$

$$m_n = -\frac{1}{2}(2a_n + 3b_n - 2), \tag{95}$$

and analogously for the parameters $\hat{l}_n$ and $\hat{m}_n$ in the hierarchy (71).

We have for the hierarchy (92) the BT (72), (73), with the corresponding shift on parameters (74), (75), i.e.

$$\hat{a}_n = -\frac{1}{4}(2a_n - 3b_n - 6), \tag{96}$$

$$\hat{b}_n = \frac{1}{2}(2a_n + b_n - 2). \tag{97}$$

In the case $n = 1$ this BT reads

$$\psi = \hat{\psi} + \frac{b_1 - 2a_1 + 2}{\hat{\psi} - 2\hat{s} - 2x}, \tag{98}$$

$$s = \hat{s} + \frac{1}{2}\frac{b_1 - 2a_1 + 2}{\hat{\psi} - 2\hat{s} - 2x}, \tag{99}$$

and eliminating $\hat{\psi}$ we obtain a BT for P_{IV} (37) itself,

$$s = \hat{s} + \frac{(2a_1 - b_1 - 2)\hat{s}}{\hat{s}_x + \hat{s}^2 + 2x\hat{s} - b_1/2 - a_1 + 1}. \tag{100}$$

This BT for P_{IV}, along with the parameter shift (96), (97) (for $n = 1$), is often refereed to as the "dagger" ($\tau^\dagger$) BT (see [38, 39]); it is for this reason that we refer to the BT (72), (73), together with the parameter shifts (96), (97), as the $\tau^\dagger$ BT for the P_{IV} hierarchy (92). Here we use the letter "t" (e.g. $t^\ddagger$) for BTs related to the hierarchy (83), and "τ" (e.g. $\tau^\dagger$) for BTs related to the hierarchy (92).

Finally we recall that we also have, as detailed in Section Four, special integrals for our P_{IV} hierarchies. Thus we have the special integral system (76)—(77), with parameters satisfying (78), for the hierarchy (83), where we now impose the identification (84)—(86). Similarly we have the special integral system (79)—(80), with parameters satisfying (81), for the hierarchy (92), now imposing (93)—(95).

In the case $n = 1$, with the identification (84)—(86), our special integral system (76)—(77) for the hierarchy (83) reads

$$\phi + 2p + 2x = 0, \tag{101}$$

$$p_x + \phi p + p^2 + B_1 = 0, \tag{102}$$

for parameters satisfying (78), i.e.

$$B_1 = 2A_1 + 2. \tag{103}$$

Thus we obtain the special integral of P_{IV} (34),

$$p_x - p^2 - 2xp + B_1 = 0, \tag{104}$$

where the parameters A_1 and B_1 of P_{IV} satisfy (103).

On the other hand, the special integral system (79)—(80) for the hierarchy (92), with the identification (93)—(95), reads for $n = 1$,

$$\psi - 2s - 2x = 0, \tag{105}$$

$$s_x + \psi s - s^2 - b_1 = 0, \tag{106}$$

with parameters satisfying (81), i.e.

$$b_1 = 2a_1 - 2. \tag{107}$$

Eliminating ψ then gives the special integral of P_{IV} (37),

$$s_x + s^2 + 2xs - b_1 = 0, \tag{108}$$

for parameters a_1 and b_1 of P_{IV} satisfying (107).

We note that the identifications of parameters (84)—(86) and (93)—(95) mean that we no longer have the simple mapping $(\phi, p, e_n, f_n) \to (\psi, -s, l_n, -m_n)$ between the hierarchies (83) and (92); consider for example the systems obtained for $n = 1$, (32), (33) and (35), (36). Thus the BTs and special integrals obtained here are no longer equivalent under this mapping. The question of whether a mapping can be found under which they are equivalent is discussed in Section Seven.

6. Further Bäcklund transformations for our P_{IV} hierarchies

Thus far we have found the BTs $t^\ddagger$ and $\tau^\dagger$ for our P_{IV} hierarchies. However, as is well known, for P_{IV} itself, these BTs can be written as compositions of other BTs, referred to in the literature [38, 39] as the "tilde" $\left(\tilde{t}/\tilde{\tau}\right)$ and "hat" $\left(\hat{t}/\hat{\tau}\right)$ BTs. We now show how this pattern of BTs for P_{IV} can be extended to our P_{IV} hierarchies.

It turns out, quite remarkably, that this can be done by considering the Miura maps between (83) and (15), and (92) and (15), as given in (11). Let us begin with the Miura transformation between (83) and (15), as given by (9). We recall that

for $n = 1$ (15) yields equation (28). We take the lower sign in (28) and now fix the relationship between our parameters c_n, d_n and α_n, β_n, for the entire hierarchy (15), as

$$g_n = 2, \tag{109}$$
$$c_n = \frac{1}{2}(2 - 2\alpha_n - \beta_n), \tag{110}$$
$$d_n = \beta_n, \tag{111}$$

and similarly for a second copy of our hierarchy (15) in $\tilde{U}$, $\tilde{V}$ with parameters $\tilde{c}_n$, $\tilde{d}_n$, or equivalently $\tilde{\alpha}_n$, $\tilde{\beta}_n$.

Since we have

$$c_n = e_n, \tag{112}$$
$$d_n = f_n - 2e_n \tag{113}$$

and similarly for parameters $\tilde{c}_n$, $\tilde{d}_n$, $\tilde{e}_n$, and $\tilde{f}_n$, we obtain the following BTs and parameter shifts:

$$\tilde{U} = \tilde{\phi} + 2\tilde{p}, \tag{114}$$
$$\tilde{V} = \tilde{p}, \tag{115}$$

with

$$\tilde{A}_n = -\frac{1}{4}(2 + 2\tilde{\alpha}_n - 3\tilde{\beta}_n), \tag{116}$$
$$\tilde{B}_n = \frac{1}{2}(2 - 2\tilde{\alpha}_n - \tilde{\beta}_n); \tag{117}$$

and

$$\phi = U - 2V, \tag{118}$$
$$p = V, \tag{119}$$

with

$$\alpha_n = \frac{1}{4}(2 - 2A_n - 3B_n), \tag{120}$$
$$\beta_n = \frac{1}{2}(2 + 2A_n - B_n). \tag{121}$$

For the case $n = 1$, the first of these, when written as a BT between two copies of P_{IV} — (34) in $\tilde{p}$, $\tilde{A}_1$, $\tilde{B}_1$, and (28) in $\tilde{y}$, $\tilde{\alpha}_1$, $\tilde{\beta}_1$, where $\tilde{y} = -\tilde{U} - 2x$ — reads

$$\tilde{y} = \frac{\tilde{p}_x - \tilde{p}^2 - 2x\tilde{p} + 1 - \tilde{\alpha}_1 - \tilde{\beta}_1/2}{2\tilde{p}}, \tag{122}$$

which, together with the parameter shifts (116), (117) defines precisely the BT $\tilde{t}$.

The second of the above BTs, in the case $n = 1$, when written as a BT between (34) in p, A_1, B_1 and (28) in y, α_1 and β_1, where $y = -U - 2x$, reads

$$p = -\frac{y_x + y^2 + 2xy + 1 + A_1 - B_1/2}{2y} \tag{123}$$

which, together with the parameter shifts (120), (121) defines precisely the BT $\hat{t}$.

We thus define the BTs (114), (115) and (118), (119), with parameter shifts (116), (117) and (120), (121) respectively, as $\tilde{t}$ and $\hat{t}$ BTs for our P_{IV} hierarchies.

We now define the additional BT S by $S = (\hat{t})^{-1} \circ t^{\ddagger} \circ (\tilde{t})^{-1}$. A simple calculation gives this BT S as

$$U = \tilde{U}, \tag{124}$$

$$V = \tilde{V} + \frac{d_n}{L_n\left[\begin{pmatrix} \tilde{U} \\ \tilde{U}\tilde{V} - \tilde{V}^2 + \tilde{V}_x \end{pmatrix}\right]}, \tag{125}$$

with the change of parameters

$$\tilde{c}_n = c_n + d_n, \tag{126}$$

$$\tilde{d}_n = -d_n, \tag{127}$$

or equivalently

$$\tilde{\alpha}_n = \alpha_n, \tag{128}$$

$$\tilde{\beta}_n = -\beta_n. \tag{129}$$

For the case $n = 1$, for equation (28), this BT S reads

$$y = \tilde{y}, \qquad \tilde{\alpha}_1 = \alpha_1, \qquad \tilde{\beta}_1 = -\beta_1, \tag{130}$$

and we recover the well known relation $t^{\ddagger} = \hat{t} \circ S \circ \tilde{t}$ for P_{IV} BTs. Our BT S (124)—(129) then allows us to extend this decomposition of the BT $t^{\ddagger}$ as $t^{\ddagger} = \hat{t} \circ S \circ \tilde{t}$ from P_{IV} itself to our P_{IV} hierarchies. This pattern of BTs, obtained here using the Miura map $\mathbf{U} = \mathbf{\Phi}[\boldsymbol{\phi}]$ of the DWW hierarchy (which defines the BTs $\hat{t}$ and $\tilde{t}$), can be seen in Figure One. It is interesting that this Miura map, a simple linear map when considered as a mapping between $\boldsymbol{\phi}$ and $\mathbf{U}$, gives rise to BTs of our hierarchies: however, as we have seen above for $n = 1$ (P_{IV}), when considered as a mapping between components of our hierarchies, it is no longer a linear map.

We recall that for $n = 1$ the second component V of the system (15) also defines a copy of P_{IV} (31). Using the identification (84)—(86), where as usual $c_1 = e_1$ and $d_1 = f_1 - 2e_1$, we obtain that equation (31), with the upper choice of sign, is a copy of equation (34), with $w = p$ and the same parameters A_1 and B_1. Thus, in Figure One, when tracing for $n = 1$ the action of our BTs over individual components, we see that the auto-BT for (34) must be the same as the auto-BT for equation (31). This last is as given by (125), and reads (with $V = w$, $\tilde{V} = \tilde{w}$, and eliminating $\tilde{U}$),

$$w = \tilde{w} + \frac{(B_1 - 2A_1 - 2)\tilde{w}}{\tilde{w}_x - \tilde{w}^2 - 2x\tilde{w} + B_1/2 + A_1 + 1}, \tag{131}$$

with parameter shifts

$$\tilde{A}_1 = -\frac{1}{4}(2A_1 - 3B_1 + 6), \tag{132}$$

$$\tilde{B}_1 = \frac{1}{2}(2A_1 + B_1 + 2). \tag{133}$$

Thus we see that this BT is exactly the same as that for (34), i.e. (91) and (87), (88), with $n = 1$. That is, for $n = 1$, (125) is the $t^{\ddagger}$ BT (from $\tilde{w}$ to w).

From the above it also follows that the equation obtained when eliminating $\tilde{U}$ from the system (29), (30) written in terms of $\tilde{U}$, $\tilde{V}$, $\tilde{c}_1$, $\tilde{d}_1$, i.e.

$$\tilde{U} = -\frac{\tilde{V}_x - \tilde{V}^2 + 2x\tilde{V} + \tilde{c}_1}{2\tilde{V}}, \tag{134}$$

corresponds to the $\tilde{t}$ BT from (31), with upper sign, to (28), with lower sign. In the same way, the equation obtained when eliminating V from the system (29), (30),

$$V = -\frac{U_x - U^2 - 2xU + 2 - d_1}{2U + 4x}, \tag{135}$$

corresponds to the $\hat{t}$ BT from (28), with lower sign, to (31), with upper sign. This then gives one identification of equations (134) and (135) (another is made later).

We now turn to our $\tau^\dagger$ BT. We recall once again the Miura maps (11) and in particular the Miura transformation between (92) and (15), as given by (10). For $n = 1$ (15) yields equation (28); we take the upper sign in (28) and change the relationship between our parameters (now labelled $\bar{c}_n$, $\bar{d}_n$ and $\bar{\alpha}_n$, $\bar{\beta}_n$, corresponding to variables $\bar{U}$, $\bar{V}$), to the following, again for the entire hierarchy (15):

$$g_n = -2, \tag{136}$$

$$\bar{c}_n = -\frac{1}{2}(2 + 2\bar{\alpha}_n + \bar{\beta}_n), \tag{137}$$

$$\bar{d}_n = \bar{\beta}_n, \tag{138}$$

and similarly for a second copy of our hierarchy (15) in $\hat{U}$, $\hat{V}$ with parameters $\hat{c}_n$, $\hat{d}_n$, or equivalently $\hat{\alpha}_n$, $\hat{\beta}_n$.

Since we have

$$\bar{c}_n = l_n, \tag{139}$$

$$\bar{d}_n = -m_n - 2l_n \tag{140}$$

and similarly for parameters $\hat{c}_n$, $\hat{d}_n$, $\hat{l}_n$, and $\hat{m}_n$, we obtain the following BTs and parameter shifts:

$$\hat{U} = \hat{\psi} - 2\hat{s}, \tag{141}$$

$$\hat{V} = -\hat{s}, \tag{142}$$

with

$$\hat{a}_n = \frac{1}{4}(2 - 2\hat{\alpha}_n + 3\hat{\beta}_n), \tag{143}$$

$$\hat{b}_n = -\frac{1}{2}(2 + 2\hat{\alpha}_n + \hat{\beta}_n); \tag{144}$$

and

$$\psi = \bar{U} - 2\bar{V}, \tag{145}$$

$$s = -\bar{V}, \tag{146}$$

with

$$\bar{\alpha}_n = -\frac{1}{4}(2 + 2a_n + 3b_n), \tag{147}$$

$$\bar{\beta}_n = -\frac{1}{2}(2 - 2a_n + b_n). \tag{148}$$

For the case $n = 1$, the first of these, when written as a BT between two copies of P_{IV} — (37) in $\hat{s}$, $\hat{a}_1$, $\hat{b}_1$, and (28) in $\hat{y}$, $\hat{\alpha}_1$, $\hat{\beta}_1$, where $\hat{y} = \hat{U} - 2x$ — reads

$$\hat{y} = -\frac{\hat{s}_x + \hat{s}^2 + 2x\hat{s} + 1 + \hat{\alpha}_1 + \hat{\beta}_1/2}{2\hat{s}}, \tag{149}$$

which, together with the parameter shifts (143), (144) defines the BT $\hat{\tau}$ (we recall, when comparing to the BT (123) with parameter shifts, (120), (121) for $n = 1$ — also identified as a "hat" BT, $\hat{t}$ — the invariance of P_{IV} (82) under $\beta \to -\beta$).

The second of the above BTs, in the case $n = 1$, when written as a BT between (37) in s, a_1, b_1 and (28) in $\bar{y}$, $\bar{\alpha}_1$ and $\bar{\beta}_1$, where $\bar{y} = \bar{U} - 2x$, reads

$$s = \frac{\bar{y}_x - \bar{y}^2 - 2x\bar{y} + 1 - a_1 + b_1/2}{2\bar{y}} \tag{150}$$

which, together with the parameter shifts (147), (148) defines the BT $\tilde{\tau}$ (again we recall, when comparing to (122) with parameter shifts, (116), (117) for $n = 1$ — also identified as a "tilde" BT, $\tilde{t}$ — the invariance of P_{IV} (82) under $\beta \to -\beta$).

We thus define the BTs (141), (142) and (145), (146), with parameter shifts (143), (144) and (147), (148) respectively, as $\hat{\tau}$ and $\tilde{\tau}$ BTs for our P_{IV} hierarchies.

We now define the additional BT σ by $\sigma = (\tilde{\tau})^{-1} \circ \tau^\dagger \circ (\hat{\tau})^{-1}$. A simple calculation gives this BT σ as

$$\bar{U} = \hat{U}, \tag{151}$$

$$\bar{V} = \hat{V} + \frac{\bar{d}_n}{L_n\left[\begin{pmatrix} \hat{U} \\ \hat{U}\hat{V} - \hat{V}^2 + \hat{V}_x \end{pmatrix}\right]}, \tag{152}$$

with the change of parameters

$$\hat{c}_n = \bar{c}_n + \bar{d}_n, \tag{153}$$

$$\hat{d}_n = -\bar{d}_n, \tag{154}$$

or equivalently

$$\hat{\alpha}_n = \bar{\alpha}_n, \tag{155}$$

$$\hat{\beta}_n = -\bar{\beta}_n. \tag{156}$$

For the case $n = 1$, for equation (28), this BT σ reads

$$\bar{y} = \hat{y}, \qquad \hat{\alpha}_1 = \bar{\alpha}_1, \qquad \hat{\beta}_1 = -\bar{\beta}_1, \tag{157}$$

and we recover the well known relation $\tau^\dagger = \tilde{\tau} \circ \sigma \circ \hat{\tau}$ for P_{IV} BTs. Our BT σ (151)—(156) then allows us to extend this decomposition of the *BT* $\tau^\dagger$ as $\tau^\dagger = \tilde{\tau} \circ \sigma \circ \hat{\tau}$ from P_{IV} itself to our P_{IV} hierarchies. This pattern of BTs, obtained here using the Miura map $\mathbf{U} = \mathbf{\Psi}[\boldsymbol{\psi}]$ of the DWW hierarchy, can be seen in Figure Two. Again we note that what is a simple linear map, when considered as a mapping between $\boldsymbol{\psi}$ and $\mathbf{U}$, gives rise to BTs of our hierarchies (the BTs $\hat{\tau}$ and $\tilde{\tau}$), but that, once again, and as we have seen for $n = 1$ (P_{IV}), when considered as a mappping between components of our hierarchies, it is no longer such a trivial mapping.

We recall that for $n = 1$ the second component $\bar{V}$ of the system (15), now written in terms of $\bar{U}$, $\bar{V}$ and with coefficients $\bar{c}_1$, $\bar{d}_1$, also defines a copy of P_{IV} via $\bar{V} = -\bar{w}$. This copy of P_{IV} is (31) in $\bar{w}$, $\bar{c}_1$ and $\bar{d}_1$, and with lower choice of sign. We thus obtain that this copy of (31), where as usual $\bar{c}_1 = l_1$ and $\bar{d}_1 = -m_1 - 2l_1$, and using the identification (93)—(95), is a copy of equation (37), with $w = s$ and the same parameters a_1 and b_1. Thus, in Figure Two, when tracing for $n = 1$ the action of our BTs over individual components, we see that the auto-BT for (37) must be the

same as the auto-BT for this copy of equation (31). This last is as given by (152), and reads (with $\bar{V} = -\bar{w}$, $\hat{V} = -\hat{w}$, and eliminating $\hat{U}$),

$$\bar{w} = \hat{w} + \frac{(2a_1 - b_1 - 2)\hat{w}}{\hat{w}_x + \hat{w}^2 + 2x\hat{w} - b_1/2 - a_1 + 1}, \tag{158}$$

with parameter shifts

$$\hat{a}_1 = -\frac{1}{4}(2a_1 - 3b_1 - 6), \tag{159}$$

$$\hat{b}_1 = \frac{1}{2}(2a_1 + b_1 - 2). \tag{160}$$

Thus we see that this BT is exactly the same as that for (37), i.e. (100) and (96), (97), with $n = 1$. That is, for $n = 1$, (152) is the $\tau^\dagger$ BT (from $\hat{w}$ to $\bar{w}$).

From the above it also follows that the equation obtained when eliminating $\hat{U}$ from the system (29), (30), written in terms of $\hat{U}$, $\hat{V}$, $\hat{c}_1$, $\hat{d}_1$, i.e.

$$\hat{U} = -\frac{\hat{V}_x - \hat{V}^2 + 2x\hat{V} + \hat{c}_1}{2\hat{V}}, \tag{161}$$

corresponds to the $\hat{\tau}$ BT from (31), with lower sign, to (28), with upper sign. In the same way, the equation obtained when eliminating $\bar{V}$ in the system (29), (30), written in terms of $\bar{U}$, $\bar{V}$, $\bar{c}_1$, $\bar{d}_1$, i.e.

$$\bar{V} = -\frac{\bar{U}_x - \bar{U}^2 - 2x\bar{U} + 2 - \bar{d}_1}{2\bar{U} + 4x}, \tag{162}$$

corresponds to the $\tilde{\tau}$ BT from (28), with upper sign, to (31), with lower sign. Thus we have a different identification of the equations (134) and (135).

We note that the transformation induced on our original variables u and v by S and σ is just the identity together with $(\alpha_n, \beta_n) \to (\alpha_n, -\beta_n)$ (a discrete symmetry of the hierarchy (18), (19)).

7. A mapping between our sequences of P_{IV} hierarchies

In this section we consider a mapping between our two different sequences of fourth Painlevé hierarchies. These two different sequences are defined by the choice $g_n = 2$ or $g_n = -2$. As we noted earlier, this then means that we no longer have the mapping $(\phi, p, e_n, f_n) \to (\psi, -s, l_n, -m_n)$ between the hierarchies (22) and (25). However, we still have another mapping between these two hierarchies.

Consider the hierarchy (22), with the identifications (84)—(86), i.e.

$$(\mathbf{H}'[\boldsymbol{\phi}])^\dagger \, \mathbf{K}_n[\mathbf{H}[\boldsymbol{\phi}]] + (e_n, f_n)^T = 0, \tag{163}$$

where ($g_n = 2$)

$$\mathbf{K}_n[\mathbf{H}[\boldsymbol{\phi}]] = \mathbf{L}_n[\mathbf{H}[\boldsymbol{\phi}]] + \sum_{i=1}^{n-1} h_i \mathbf{L}_i[\mathbf{H}[\boldsymbol{\phi}]] + 2\begin{pmatrix} 0 \\ x \end{pmatrix}. \tag{164}$$

We now consider the scaling transformation $(\phi, p, x) = (\lambda^{-1}\psi, -\lambda^{-1}s, \lambda\xi)$. This then gives

$$\mathbf{L}_i[\mathbf{H}[\boldsymbol{\phi}]] = \begin{pmatrix} \frac{1}{\lambda^{i+1}} & 0 \\ 0 & \frac{1}{\lambda^i} \end{pmatrix} \mathbf{L}_i[\mathbf{I}[\boldsymbol{\psi}]], \tag{165}$$

where in the right-hand-side of this last, derivatives are w.r.t. ξ. Thus

$$\begin{aligned}\mathbf{K}_n[\mathbf{H}[\boldsymbol{\phi}]] &= \left(\begin{array}{cc}\frac{1}{\lambda} & 0\\ 0 & 1\end{array}\right)\frac{1}{\lambda^n}\left[\mathbf{L}_n[\mathbf{I}[\boldsymbol{\psi}]]+\sum_{i=1}^{n-1}h_i\lambda^{n-i}\mathbf{L}_i[\mathbf{I}[\boldsymbol{\psi}]]+2\lambda^{n+1}\left(\begin{array}{c}0\\ \xi\end{array}\right)\right]\\ &= \left(\begin{array}{cc}\frac{1}{\lambda} & 0\\ 0 & 1\end{array}\right)\frac{1}{\lambda^n}\left[\mathbf{L}_n[\mathbf{I}[\boldsymbol{\psi}]]+\sum_{i=1}^{n-1}H_i\mathbf{L}_i[\mathbf{I}[\boldsymbol{\psi}]]-2\left(\begin{array}{c}0\\ \xi\end{array}\right)\right],\end{aligned} \tag{166}$$

where we have chosen λ such that $\lambda^{n+1}=-1$, and have set $h_i\lambda^{n-i}=H_i$. Since also

$$(\mathbf{H}'[\boldsymbol{\phi}])^\dagger = \left(\begin{array}{cc}1 & 0\\ 0 & -1\end{array}\right)(\mathbf{I}'[\boldsymbol{\psi}])^\dagger\left(\begin{array}{cc}1 & 0\\ 0 & \frac{1}{\lambda}\end{array}\right), \tag{167}$$

our hierarchy (22), i.e. (163), becomes

$$(\mathbf{I}'[\boldsymbol{\psi}])^\dagger\left[\mathbf{L}_n[\mathbf{I}[\boldsymbol{\psi}]]+\sum_{i=1}^{n-1}H_i\mathbf{L}_i[\mathbf{I}[\boldsymbol{\psi}]]-2\left(\begin{array}{c}0\\ \xi\end{array}\right)\right]+\left(\begin{array}{c}-e_n\\ f_n\end{array}\right)=\left(\begin{array}{c}0\\ 0\end{array}\right), \tag{168}$$

which, identifying $e_n=-l_n$ and $f_n=m_n$, is precisely the hierarchy (25), i.e.

$$(\mathbf{I}'[\boldsymbol{\psi}])^\dagger\,\mathbf{K}_n[\mathbf{I}[\boldsymbol{\psi}]]+(l_n,m_n)^T=0, \tag{169}$$

with the identification (93)—(95) ($g_n=-2$). Thus we have a BT from the hierarchy (25) with $g_n=-2$ to the hierarchy (22) with $g_n=2$, given by

$$(\phi,p,x) = (\lambda^{-1}\psi,-\lambda^{-1}s,\lambda\xi), \tag{170}$$

$$l_n = -e_n, \tag{171}$$

$$m_n = f_n, \tag{172}$$

$$H_i = h_i\lambda^{n-i}, \tag{173}$$

where $\lambda^{n+1}=-1$. We refer to this BT as the transformation T. We now consider the composition $T^{-1}\circ t^\ddagger\circ T$, i.e. $(\hat{\psi},\hat{s},\hat{l}_n,\hat{m}_n)\to(\tilde{\phi},\tilde{p},\tilde{e}_n,\tilde{f}_n)\to(\phi,p,e_n,f_n)\to(\psi,s,l_n,m_n)$, which gives an auto-BT of the hierarchy (25). It turns out that this auto-BT is precisely $\tau^\dagger$. That is, we have the relation

$$\tau^\dagger = T^{-1}\circ t^\ddagger\circ T. \tag{174}$$

We note that T also maps the first integrals (79), (80) of the hierarchy (25), with the identifications (93)—(95) ($g_n=-2$) to the first integrals (76), (77) of the hierarchy (22), with the identifications (84)—(86) ($g_n=2$).

We now consider a corresponding scaling transformation between our hierarchy (15) for $g_n=2$, and the same hierarchy for $g_n=-2$. This scaling transformation is induced from that between the hierarchies (22) and (25) as $(U,V,x)=(\lambda^{-1}\bar{U},\lambda^{-1}\bar{V},\lambda\xi)$. The hierarchy (15), i.e.

$$(\mathbf{F}'[\mathbf{U}])^\dagger\,\mathbf{K}_n[\mathbf{F}[\mathbf{U}]]+(c_n,d_n)^T=0, \tag{175}$$

where

$$\mathbf{K}_n[\mathbf{F}[\mathbf{U}]]=\mathbf{L}_n[\mathbf{F}[\mathbf{U}]]+\sum_{i=1}^{n-1}h_i\mathbf{L}_i[\mathbf{F}[\mathbf{U}]]+2\left(\begin{array}{c}0\\ x\end{array}\right), \tag{176}$$

then becomes (again all derivatives are now with respect to ξ)

$$\left(\mathbf{F}'[\bar{\mathbf{U}}]\right)^\dagger\,\mathbf{K}_n[\mathbf{F}[\bar{\mathbf{U}}]]+(\bar{c}_n,\bar{d}_n)^T=0, \tag{177}$$

with

$$\mathbf{K}_n[\mathbf{F}[\bar{\mathbf{U}}]] = \mathbf{L}_n[\mathbf{F}[\bar{\mathbf{U}}]] + \sum_{i=1}^{n-1} H_i \mathbf{L}_i[\mathbf{F}[\bar{\mathbf{U}}]] - 2\begin{pmatrix} 0 \\ \xi \end{pmatrix}, \tag{178}$$

and where we have identified $c_n = -\bar{c}_n$ and $d_n = -\bar{d}_n$. That is, we have the BT from (177), (178) to (175), (176),

$$\begin{aligned} (U, V, x) &= (\lambda^{-1}\bar{U}, \lambda^{-1}\bar{V}, \lambda\xi), && (179)\\ \bar{c}_n &= -c_n, && (180)\\ \bar{d}_n &= -d_n, && (181)\\ H_i &= h_i\lambda^{n-i}, && (182) \end{aligned}$$

with $\lambda^{n+1} = -1$, which, by a convenient abuse of notation, we also refer to as the transformation T.

We now consider the composition $T^{-1}\circ\tilde{t}\circ T$, i.e. $(\hat{\psi}, \hat{s}, \hat{l}_n, \hat{m}_n) \to (\tilde{\phi}, \tilde{p}, \tilde{e}_n, \tilde{f}_n) \to (\tilde{U}, \tilde{V}, \tilde{c}_n, \tilde{d}_n) \to (\hat{U}, \hat{V}, \hat{c}_n, \hat{d}_n)$. This BT turns out to be precisely the BT $\hat{\tau}$. Thus we have the relation

$$\hat{\tau} = T^{-1}\circ\tilde{t}\circ T. \tag{183}$$

We also consider the composition $T^{-1}\circ\hat{t}\circ T$, i.e. $(\bar{U}, \bar{V}, \bar{c}_n, \bar{d}_n) \to (U, V, c_n, d_n) \to (\phi, p, e_n, f_n) \to (\psi, s, l_n, m_n)$. This BT is $\tilde{\tau}$, and so we have

$$\tilde{\tau} = T^{-1}\circ\hat{t}\circ T. \tag{184}$$

Finally, consideration of the BT $T^{-1}\circ S\circ T$, i.e. $(\hat{U}, \hat{V}, \hat{c}_n, \hat{d}_n) \to (\tilde{U}, \tilde{V}, \tilde{c}_n, \tilde{d}_n) \to (U, V, c_n, d_n) \to (\bar{U}, \bar{V}, \bar{c}_n, \bar{d}_n)$, leads to the conclusion

$$\sigma = T^{-1}\circ S\circ T. \tag{185}$$

Thus we see that our transformation T is a mapping of Figure Two into Figure One, but for a different independent variable, x in Figure One being related to ξ in figure Two by $x = \lambda\xi$ where $\lambda^{n+1} = -1$. Thus of course the relation $\tau^{\dagger} = \tilde{\tau}\circ\sigma\circ\hat{\tau}$ (Figure Two) is mapped into the relation $t^{\ddagger} = \hat{t}\circ S\circ\tilde{t}$ (Figure One).

Our transformation T has some important consequences. It tells us that the pattern of BTs obtained from our second sequence of Painlevé hierarchies can be related to that obtained from our first sequence of Painlevé hierarchies. If we had only considered one sequence (e.g. the first) it might not have been obvious how to obtain a sequence (the second) having the pattern of BTs corresponding to $\tau^{\dagger}$.

The reason why this might not have been obvious is that, for P_{IV}, the BTs "tilde" and 'hat" are believed to be independent. This then leads us on to another of the important consequences of our results: the BTs "tilde" and 'hat" for P_{IV} are not independent, but are related by a trivial scaling of P_{IV}. That is, *there is only one nontrivial fundamental BT for P_{IV}*. This is in contrast to the claim in [38] that P_{IV} has two nontrivial fundamental BTs ("tilde" and 'hat").

Let us present our results for P_{IV} explicitly. For $n = 1$ we may take $\lambda = i$ and so our transformation T from (37) with the identification (93)—(95),

$$s_{\xi\xi} = \frac{1}{2}\frac{s_\xi^2}{s} + \frac{3}{2}s^3 + 4\xi s^2 + 2\left[\xi^2 - a_1\right]s - \frac{1}{2}\frac{b_1^2}{s}, \tag{186}$$

to (34) with the identification (84)—(86),

$$p_{xx} = \frac{1}{2}\frac{p_x^2}{p} + \frac{3}{2}p^3 + 4xp^2 + 2\left[x^2 - A_1\right]p - \frac{1}{2}\frac{B_1^2}{p}, \tag{187}$$

is

$$p = is, \qquad x = i\xi, \qquad a_1 = -A_1 \qquad b_1 = -B_1. \tag{188}$$

The same transformation T provides a BT from (28) in $\bar{y}$ and ξ, with upper choice of sign and the identification (136)—(138),

$$\bar{y}_{\xi\xi} = \frac{1}{2}\frac{\bar{y}_\xi^2}{y} + \frac{3}{2}\bar{y}^3 + 4\xi\bar{y}^2 + 2\left(\xi^2 - \bar{\alpha}_1\right)\bar{y} - \frac{1}{2}\frac{\bar{\beta}_1^2}{\bar{y}}, \tag{189}$$

to (28) in y and x, with lower choice of sign and the identification (109)—(111),

$$y_{xx} = \frac{1}{2}\frac{y_x^2}{y} + \frac{3}{2}y^3 + 4xy^2 + 2\left(x^2 - \alpha_1\right)y - \frac{1}{2}\frac{\beta_1^2}{y}, \tag{190}$$

i.e.

$$y = i\bar{y}, \qquad x = i\xi, \qquad \bar{\alpha}_1 = -\alpha_1 \qquad \bar{\beta}_1 = -\beta_1. \tag{191}$$

In order to show explicitly that P_{IV} has only one fundamental BT it is enough to show that (184) holds, i.e. that

$$\tilde{\tau} = T^{-1} \circ \hat{t} \circ T. \tag{192}$$

Here $\hat{t}$ is the BT (123), with parameter shifts (120) and (121), i.e.

$$p = -\frac{y_x + y^2 + 2xy + 1 + A_1 - B_1/2}{2y}, \tag{193}$$

and

$$\alpha_1 = \frac{1}{4}(2 - 2A_1 - 3B_1), \tag{194}$$

$$\beta_1 = \frac{1}{2}(2 + 2A_1 - B_1), \tag{195}$$

from (190) to (187). Meanwhile, $\tilde{\tau}$ is the BT (150), with parameter shifts (147), (148), i.e.

$$s = \frac{\bar{y}_\xi - \bar{y}^2 - 2\xi\bar{y} + 1 - a_1 + b_1/2}{2\bar{y}} \tag{196}$$

$$\bar{\alpha}_1 = -\frac{1}{4}(2 + 2a_1 + 3b_1), \tag{197}$$

$$\bar{\beta}_1 = -\frac{1}{2}(2 - 2a_1 + b_1), \tag{198}$$

from (189) to (186). It is easy to show that under the transformation T, i.e. when (188) and (191) hold, equations (196)—(198) are mapped onto equations (193)—(195). Thus the "tilde" and "hat" BTs of P_{IV} are equivalent under a simple scaling transformation, and we see that P_{IV} has only one nontrivial fundamental auto-BT.

We note that for P_{IV} itself the transformation T can in fact be found in [40], and was also known to the authors of [38]. However these last failed to recognize that it provides a mapping between the "tilde" and "hat" BTs of P_{IV}.

8. Conclusions

We have given an improved method of obtaining auto-BTs and special integrals for hierarchies of ODEs, and have used this to derive auto-BTs and special integrals for two fourth Painlevé hierarchies. We have shown how the known pattern of BTs for P_{IV} can be extended to hierarchies, observing that the BTs required to do this turn out to be precisely the Miura maps of the DWW hierarchy. Finally, we have given a mapping between our two sequences of fourth Painlevé hierarchies which allows us to relate the BTs derived for these two sequences: in particular, we have derived the result that P_{IV} has in fact only one nontrivial fundamental auto-BT.

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$$
\begin{tikzcd}
\begin{pmatrix} c_n \\ d_n \end{pmatrix} \sim \begin{pmatrix} \alpha_n \\ \beta_n \end{pmatrix} & & \begin{pmatrix} \tilde{c}_n \\ \tilde{d}_n \end{pmatrix} \sim \begin{pmatrix} \tilde{\alpha}_n \\ \tilde{\beta}_n \end{pmatrix} \\
\begin{pmatrix} U \\ V \end{pmatrix} \arrow[d, "\hat{t}"'] & & \begin{pmatrix} \tilde{U} \\ \tilde{V} \end{pmatrix} \arrow[ll, "s"'] \\
\begin{pmatrix} \phi \\ p \end{pmatrix} & & \begin{pmatrix} \tilde{\phi} \\ \tilde{p} \end{pmatrix} \arrow[ll, "t^{\ddagger}"] \arrow[u, "\tilde{t}"'] \\
\begin{pmatrix} e_n \\ f_n \end{pmatrix} \sim \begin{pmatrix} A_n \\ B_n \end{pmatrix} & & \begin{pmatrix} \tilde{e}_n \\ \tilde{f}_n \end{pmatrix} \sim \begin{pmatrix} \tilde{A}_n \\ \tilde{B}_n \end{pmatrix}
\end{tikzcd}
$$

Figure One: Decomposition of the BT $t^{\ddagger}$ for P_{IV} hierarchies ($g_n = 2$).

$$
\begin{tikzcd}
\begin{pmatrix} \bar{c}_n \\ \bar{d}_n \end{pmatrix} \sim \begin{pmatrix} \bar{\alpha}_n \\ \bar{\beta}_n \end{pmatrix} & & \begin{pmatrix} \hat{c}_n \\ \hat{d}_n \end{pmatrix} \sim \begin{pmatrix} \hat{\alpha}_n \\ \hat{\beta}_n \end{pmatrix} \\
\begin{pmatrix} \bar{U} \\ \bar{V} \end{pmatrix} \arrow[d, "\tilde{\tau}"'] & & \begin{pmatrix} \hat{U} \\ \hat{V} \end{pmatrix} \arrow[ll, "\sigma"'] \\
\begin{pmatrix} \psi \\ s \end{pmatrix} & & \begin{pmatrix} \hat{\psi} \\ \hat{s} \end{pmatrix} \arrow[ll, "\tau^{\dagger}"] \arrow[u, "\hat{\tau}"'] \\
\begin{pmatrix} l_n \\ m_n \end{pmatrix} \sim \begin{pmatrix} a_n \\ b_n \end{pmatrix} & & \begin{pmatrix} \hat{l}_n \\ \hat{m}_n \end{pmatrix} \sim \begin{pmatrix} \hat{a}_n \\ \hat{b}_n \end{pmatrix}
\end{tikzcd}
$$

Figure Two: Decomposition of the BT $\tau^{\dagger}$ for P_{IV} hierarchies ($g_n = -2$).